

The non-linear theory of a warped accretion disc with the β -viscosity prescription

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Abstract. We study the nonlinear dynamics of a warped or twisted accretion disc, in which the viscosity coefficients are assumed to be locally proportional to the rotational velocity (β -prescription). Using asymptotic methods for thin discs, dynamical equations of the disc are obtained in a warped spherical polar coordinates. These equations are solved by the method of the separation of the variables. This analysis constitutes an analogous study of the nonlinear theory of an alpha model warped disc which has been studied by Ogilvie (1999). We have compared our results with Ogilvie’s analysis. The dynamical behaviors of these models have also been discussed. Our results show that different viscosity prescriptions and magnitudes (α and β prescriptions) affect dynamics of a warped accretion disc. Therefore it can be important in determining the viscosity law even for a warped disc.

Keywords : accretion, accretion discs – hydrodynamics

1. Introduction

The development of studies about accretion discs during present century is an illustration of the grow of interest of researches. In more recent times a good deal of attention has been devoted to studies of the warped accretion discs. This kind of discs has been observed in a wide variety of astronomical objects from young stellar objects, X-ray binary stars to active galactic nuclei. For example, the stability of the super orbital periodicity in the

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neutron star XRBs Cyg X-2, LMC X-4 and Her X-1 (Clarkson et al. 2003) is attributed to a warped disc. The outflows and illumination patterns from the central engine of a Seyfert Galaxy (Greenhill et al. 2003) provide direct reasoning to warped accretion discs. Also, observations of FUSE (Far Ultraviolet Spectroscopic Explorer) and HST (Hubble Space Telescope) ultraviolet of the low-inclination, nova-like Cataclysmic variable RW Sex presented evidences of warped discs (Prinja et al. 2003). The precession of warped discs in magnetized stars (T Tauri stars, white dwarf or neutron stars) discussed by Pfeiffer & Lai (2004) that can be concluded from magnetic torques due to the interaction between the stellar field and the induced electric currents in the disc. A photometric study of the SW Sex star, PX And (Boffin et al. 2003) reveals a precessing disc possibly warped. Many different systems, including young stellar objects and X-ray binaries, display a different precession period or radiation flux variabilities that it may be concluded from a warped accretion disc. Also, Her X-1 (Tananbaum et al. 1972; Katz 1973; Roberts 1974; Still & Boyd 2004) and the hard X-ray component of the micro quasar GRS1915+105 (Rau, Greiner & McCollough 2003) show another typical behavior of warped discs.

During recent years, several driving mechanisms for disc warps have been suggested. A warp may be induced through instabilities that may be due to viscous torque (Pringle 1992), bending waves and viscous torques (Ogilvie 1999), resonant tidal interactions (Lubow 1992; Lubow & Ogilvie 2000); irradiation-driven wind torques (Schandl & Meyer 1994); radiation torques (Pringle 1996); wind torques via Kelvin-Helmholts instability (Quillen 2001); magnetic torques (Lai 1999) and magnetic-induced electric current torques (Lai 2003).

Most theoretical studies of the accretion discs are based on the concept of a real fluid and the equations of magnetohydrodynamics. Adjacent layers of a moving real fluid experience tangential forces (shearing stresses) as well as normal forces (pressures). It seems that the viscous forces are the main physical agent in any accretion disc theory. Therefore, the viscous forces are parameterized. A way of doing this is through the dimensionless parameter α introduced by Shakura & Sunyaev (1973). It describes angular momentum transport in accretion discs. The mechanisms of the angular momentum transport can be described by the magnetic or the hydrodynamic instabilities. There is another prescription through the dimensionless parameter β introduced by Lynden-Bell & Pringle (1974), in which the viscosity coefficient is proportional to the rotational velocity. Analyzing the turbulent flows between the coaxial cylinders, Richard & Zahn (1999) investigated β -prescription for turbulent viscosity and found that the parameter β is of the order of 10^{-5} . Furthermore, in their model, the viscosity is very small comparing with the viscosity in α -prescription. Consequently, the forces due to viscous friction are very small compared with the remaining forces (gravity and pressure forces). So, the Reynolds numbers are very large, because of very low viscosity of the fluid. Accretion disc model with β -prescription has been studied by Hure, Richard & Zahn (2001). They argued that the beta model for viscosity prescription is applicable for analyzing the steady state structure of the Keplerian accretion discs and it yields somewhat different results compared to the classical α -viscosity introduced by Shakura & Sunyaev (1973). Study

on hydrodynamic viscosity and selfgravitation in non-warped accretion discs (β model) has been done by Duschl et al.(2000). They showed that β -discs can explain the observed spectra of protoplanetary discs and yield a natural solution to an inconsistency in the α -disc models if the mass of the disc is large enough for self-gravity to play a role and in the limit of low mass, hydrodynamic turbulence will result by α model. Turbulence induced by the horizontal and vertical shear has been studied by Mathis et al. (2004). They have presented a new prescription, the β -viscosity, for the horizontal component of the turbulent diffusivity due to the differential rotation in latitude. They generalized β prescription (Richard & Zahn 1999) in the stationery limit, advection and diffusion balance each other. Because that prescription (Richard & Zahn 1999) has been established in the case of maximum differential rotation and so its validity must be verified to milder shear rates. They have examined various prescriptions with their work. Their prescription yields a better agreement with the observations, but one can hardly consider it as the final answer, especially for extreme differential rotation.

In general, what we can find from previous works on accretion discs is the effects of various viscosity prescriptions on the structure of discs and under conditions, their results are compatible to some prescriptions. However, almost all previous studies of β -discs dedicated to non-warped accretion discs.

In order to investigate the non-linear dynamics of warped accretion discs and whether selecting of the model affects the forms of the equations governing a warped viscous disc, we applied Ogilvie's method (1999, hereafter OG) as considered in a α theory and check this new prescription for a thin viscous disc. Our study includes numerical solutions of the Navier-Stokes equations. First, the problem is reduced to a so-called singular perturbation which is then solved by the method of matched asymptotic expansions.

The structure of the paper is as the following. In the next section, we explain general formulation. Analysis of the problem is presented in section 3. Then we show that the equations can be solved using the method of the separation of the variables and the numerical solutions are discussed in section 4. We compare the warped α and β -discs in section 5 and then we conclude in section 6.

2. General Formulation

In order to construct a model for a warped accretion disc, we start by writing hydrodynamic equations. Then, we use the appropriate forms of these fundamental equations in warped spherical polar coordinates (r, θ, ϕ) (OG). Although many physical agents such as magnetic fields and radiative processes are playing significant roles in the dynamics of the discs, we neglect all those complex phenomena in order to understand the dynamics of an accretion disc with β -prescription for viscosity via semi-analytical methods. Also the self-gravity of the disc and interaction of the stellar magnetic field with the disc are

neglected. The fundamental governing equations are the continuity,

$$D\rho = -\rho\nabla \cdot \mathbf{u}, \quad (1)$$

the adiabatic condition,

$$Dp = -\Gamma p\nabla \cdot \mathbf{u}, \quad (2)$$

and finally the equation of motion,

$$\rho D\mathbf{u} = -\nabla p - \rho\nabla\Phi + \nabla \cdot [\mu\nabla\mathbf{u} + \mu(\nabla\mathbf{u})^T] + \nabla [(\mu_b - \frac{2}{3}\mu)\nabla \cdot \mathbf{u}], \quad (3)$$

where \mathbf{u} , ρ , p and Φ are the absolute velocity, the density, the pressure and the external gravitational potential, respectively. Also, Γ is the adiabatic exponent and the shear and the bulk viscosities are denoted by μ and μ_b . The symbol D denotes the Lagrangian time derivative operator,

$$D = (\partial_t)_{r,\theta,\phi} + v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi. \quad (4)$$

Note that the components (u_r, u_θ, u_ϕ) are the absolute velocity components, i.e. \mathbf{u} is the velocity as measured in the inertial frame. But the additional motion with respect to the warped coordinate system is described by the relative velocity \mathbf{v} . The exact relationship between these two velocities has been found by OG.

Warped spherical polar coordinates (r, θ, ϕ) is defined so that on each sphere $r = \text{constant}$, one can define the usual angular coordinates (θ, ϕ) , but with respect to an axis that is tilted to a point in the direction of unit vector $\ell(r, t)$. This tilt vector can be described using the Euler angles $\beta_E(r, t)$ and $\gamma(r, t)$:

$$\ell = \sin\beta_E \cos\gamma \mathbf{e}_x + \sin\beta_E \sin\gamma \mathbf{e}_y + \cos\beta_E \mathbf{e}_z. \quad (5)$$

For the viscosity coefficients, we are using the β - prescription (Lynden-Bell and Pringle 1974) rather than the usual α - prescription which has been used by OG. Thus, the viscosity coefficients are assumed to be locally proportional to the rotational velocity,

$$\mu = \beta r^2 \Omega \rho, \quad (6)$$

$$\mu_b = \beta_b r^2 \Omega \rho, \quad (7)$$

where the dimensionless coefficients β and β_b can be considered as functions of the radius. We will show that β - prescription for the viscosity coefficients lead to significant changes in the equations describing a warped disc. Also the fluid is assumed to be locally polytropic,

$$p = k\rho^\Gamma. \quad (8)$$

where $\Gamma(r)$ is a prescribed function of radius.

Now we can consider a thin disc in a spherically potential $\Phi(r)$, in which the small

parameter ϵ is a characteristic value of the local angular semi-thickness of the disc. Using this small parameter, it is possible to study the structure of a warped disc by asymptotic expansion method. In thin disc approximation, one can assume that the disc matter lies close to $\theta = \pi/2$. To resolve the internal structure of the disc, introduce the scaled dimensionless vertical coordinate ζ ,

$$\theta = \frac{\pi}{2} - \epsilon\zeta, \quad (9)$$

and the slow time coordinate,

$$T = \epsilon^2 t. \quad (10)$$

For the density and pressure, introduce the scalings

$$\rho(r, \theta, \phi, t) = \epsilon^s [\rho_0(r, \phi, \zeta, T) + \epsilon \rho_1(r, \phi, \zeta, T) + O(\epsilon^2)], \quad (11)$$

$$p(r, \theta, \phi, t) = \epsilon^{s+2} [p_0(r, \phi, \zeta, T) + \epsilon p_1(r, \phi, \zeta, T) + O(\epsilon^2)], \quad (12)$$

and for the relative velocities,

$$v_r(r, \theta, \phi, t) = \epsilon v_{r1}(r, \phi, \zeta, T) + \epsilon^2 v_{r2}(r, \phi, \zeta, T) + O(\epsilon^3), \quad (13)$$

$$v_\theta(r, \theta, \phi, t) = \epsilon v_{\theta1}(r, \phi, \zeta, T) + \epsilon^2 v_{\theta2}(r, \phi, \zeta, T) + O(\epsilon^3), \quad (14)$$

$$v_\phi(r, \theta, \phi, t) = r\Omega(r) \sin \theta + \epsilon v_{\phi1}(r, \phi, \zeta, T) + \epsilon^2 v_{\phi2}(r, \phi, \zeta, T) + O(\epsilon^3). \quad (15)$$

Finally, for the viscosities, assume

$$\mu(r, \theta, \phi, t) = \epsilon^{s+2} [\mu_0(r, \phi, \zeta, T) + \epsilon \mu_1(r, \phi, \zeta, T) + O(\epsilon^2)], \quad (16)$$

$$\mu_b(r, \theta, \phi, t) = \epsilon^{s+2} [\mu_{b0}(r, \phi, \zeta, T) + \epsilon \mu_{b1}(r, \phi, \zeta, T) + O(\epsilon^2)]. \quad (17)$$

where s is a parameter which should be positive if the self-gravitation of the disc is to be negligible. The equations of fluid dynamics were derived in warped spherical polar coordinates by OG. He reduced them by means of above asymptotic expansions for a thin disc and then divided them into two sets. Set A, which determines the intermediate velocities, consists of five coupled non-linear partial differential equations (PDEs) in two dimensions (ϕ, ζ) and seven dependent variables $\{\rho_0, p_0, \mu_0, \mu_{b0}, v_{r1}, v_{\theta1}, v_{\phi1}\}$. Also Set B, which determines the slow velocities, contains a set of five linear PDEs for the higher-order quantities $\{\rho_1, p_1, \mu_1, \mu_{b1}, v_{r2}, v_{\theta2}, v_{\phi2}\}$, including coefficients that depend upon the solutions of Set A and their radial derivatives. In the present work, we adopted Set A and B for our analysis (see OG, for details of deriving the expansions of Set A and Set B).

3. Analysis

It is very unlikely that the equations of Set A can be solved analytically. Numerical approach is a convenient way. Fortunately, we can transform the partial differential

equations into a set of ordinary differential equations (ODEs) using the method of separation of variables. To achieve this, at first we introduce the following forms for the variables:

$$h_0 = r^2 \Omega^2 [f_1(\phi - \chi) - \frac{1}{2} f_2(\phi - \chi) \zeta^2], \quad (18)$$

$$v_{r1} = r \Omega f_3(\phi - \chi) \zeta, \quad (19)$$

$$v_{\theta 1} = r \Omega [f_4(\phi - \chi) \zeta + g_4(\phi - \chi) \zeta^3], \quad (20)$$

$$v_{\phi 1} + r v_{r1} \gamma' \cos \beta_E = r \Omega [f_5(\phi - \chi) \zeta + g_5(\phi - \chi) \zeta^3]. \quad (21)$$

where $h_0 = \frac{\Gamma}{\Gamma-1} \frac{p_0}{\rho_0}$ is the enthalpy. Using the relation (18), one can drive the upper surface of the disc as

$$\zeta^2 = 2f_1(\phi - \chi) f_2^{-1}(\phi - \chi) \quad (22)$$

Introducing by the dimensionless functions f_1, \dots, f_5 and g_4, g_5 , one can obtain the coupled sets of the non-linear ODEs of first order for the given equations

$$f_1'(\phi) = (\Gamma - 1) f_4(\phi) f_1(\phi), \quad (23)$$

$$f_2'(\phi) = (\Gamma + 1) f_4(\phi) f_2(\phi) - 6(\Gamma - 1) g_4(\phi) f_1(\phi), \quad (24)$$

$$f_3'(\phi) = f_4(\phi) f_3(\phi) + 2f_5(\phi) + [f_2(\phi) - 6(\beta_b + \frac{1}{3}\beta) g_4(\phi)] |\psi| \cos \phi \quad (25)$$

$$f_4'(\phi) = -f_3'(\phi) |\psi| \cos \phi + 2f_3(\phi) |\psi| \sin \phi + f_4(\phi) [f_4(\phi) + f_3(\phi) |\psi| \cos \phi] + 1 - f_2(\phi) + 6(\beta_b + \frac{1}{3}\beta) g_4(\phi) + 6\beta g_4(\phi) (1 + |\psi|^2 \cos^2 \phi), \quad (26)$$

$$g_4'(\phi) = g_4(\phi) f_3(\phi) |\psi| \cos \phi + 4g_4(\phi) f_4(\phi), \quad (27)$$

$$f_5'(\phi) = f_4(\phi) f_5(\phi) - \frac{1}{2} \tilde{\kappa}^2 f_3(\phi) + 6\beta g_5(\phi) (1 + |\psi|^2 \cos^2 \phi), \quad (28)$$

$$g_5'(\phi) = 3f_4(\phi) g_5(\phi) + g_4(\phi) f_5(\phi), \quad (29)$$

where these functions are subject to periodic boundary conditions $f_n(2\pi) = f_n(0)$ and $g_n(2\pi) = g_n(0)$. The epicyclic frequency $\kappa(r)$ is defined by

$$\kappa^2 = 4\Omega^2 + 2r\Omega\Omega', \quad (30)$$

and the dimensionless epicyclic frequency is $\tilde{\kappa} = \kappa/\Omega$. Meanwhile the amplitude of the warp is defined as

$$|\psi| = r \left| \frac{\partial \ell}{\partial r} \right|, \quad (31)$$

hence, using the tilt vector, we have

$$\psi = |\psi| e^{i\chi} = r(\beta_E' + i\gamma' \sin \beta_E), \quad (32)$$

that it is a dimensionless complex variable.¹ Moreover, by the definition of f_6 (see, Appendix B), one can drive

$$f_6'(\phi) = -2f_4(\phi) f_6(\phi) + 9(\Gamma - 1) f_1(\phi) f_2(\phi)^{-1} g_4(\phi) f_4(\phi). \quad (33)$$

¹Throughout this paper, a prime and a dot for $\Omega(r), \beta_E$ and γ imply radial and time derivatives, respectively.

Also in addition to these, we find the following combinations

$$f_2(\phi)g_4(\phi) = 0 \quad \text{if } \Gamma \neq 1/3, \quad (34)$$

$$g_4(\phi)f_3(\phi) = -2g_5(\phi) \quad \text{all } \Gamma. \quad (35)$$

All information about a warped disc with β -prescription can be obtained by solving equations (23)-(33) numerically. Before presenting the results of numerical integration, we discuss about evolutionary equations of a warped β -disc and its relations with solutions described by functions f_n and g_n . In particular, it is very important to see whether the equations of Pringle (1992) can be derived from the three dimensional fluid equations. Pringle (1992) developed an approach for deriving the equations of a warped disc, without reference to the detailed internal fluid equations. OG showed the impossibility of deriving the angular momentum equation of Pringle (1992) from the basic equations of fluid dynamics. In fact, one should allow for a more general form of the torque between neighboring rings. Thus, OG introduced three dimensionless coefficients Q_1 , Q_2 and Q_3 , which depend on physical quantities of the disc. These coefficients enable us to discuss about relative importance of torques between neighboring rings in a warped α -disc. We are following similar approach for describing a warped β -disc. However, as we will see, in a warped β -disc there are various torques and so we should introduce extra dimensionless coefficients. Using the equations that can be extracted from Set B in OG and substituting the defined quantities $\{h_0, v_{r1}, v_{\theta1}, v_{\phi1}\}$, we can propose the evolutionary equations for the warped β -disc as (for details, see Appendix B),

$$\begin{aligned} \frac{\partial}{\partial t} (\Sigma r^2 \Omega \ell) + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma \bar{v}_r r^3 \Omega \ell) = \\ \frac{1}{r} \frac{\partial}{\partial r} (Q_1 \mathcal{I} r^2 \Omega^2 \ell) + \frac{1}{r} \frac{\partial}{\partial r} \left(Q_2 \mathcal{I} r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} (Q_3 \mathcal{I} r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r}) \\ + \frac{1}{r} \frac{\partial}{\partial r} (Q'_1 \Sigma r^4 \Omega^2 \ell) + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_2 \Sigma r^5 \Omega^2 \frac{\partial \ell}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_3 \Sigma r^5 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right), \end{aligned} \quad (36)$$

for the angular momentum, and

$$\Sigma \bar{v}_r \frac{\partial}{\partial r} (r^2 \Omega) = \frac{1}{r} \frac{\partial}{\partial r} (Q_1 \mathcal{I} r^2 \Omega^2) - Q_2 \mathcal{I} r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 + \frac{1}{r} \frac{\partial}{\partial r} (Q'_1 \Sigma r^4 \Omega^2) - Q'_2 \Sigma r^4 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2, \quad (37)$$

for the component of angular momentum parallel to ℓ , and

$$\begin{aligned} \Sigma r^2 \Omega \left(\frac{\partial \ell}{\partial t} + \bar{v}_r \frac{\partial \ell}{\partial r} \right) = \\ Q_1 \mathcal{I} r \Omega^2 \frac{\partial \ell}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left(Q_2 \mathcal{I} r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right) + Q_2 \mathcal{I} r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 \ell + \frac{1}{r} \frac{\partial}{\partial r} \left(Q_3 \mathcal{I} r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right) \\ + Q'_1 \Sigma r^3 \Omega^2 \frac{\partial \ell}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_2 \Sigma r^5 \Omega^2 \frac{\partial \ell}{\partial r} \right) + Q'_2 \Sigma r^4 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 \ell + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_3 \Sigma r^5 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right). \end{aligned} \quad (38)$$

for the tilt vector.

Thus, we can obtain coefficients Q_n and Q'_n {i.e., $n = 1, 4$ } as (Appendix B),

$$Q_1 = \langle f_6(-f_3 f_5 + 3\beta g_5 |\psi| \cos \phi) \rangle, \quad (39)$$

$$Q'_1 = \langle -\frac{1}{2}(4 - \kappa^2)\beta + \beta f_5 |\psi| \cos \phi \rangle, \quad (40)$$

$$Q_4 = \frac{1}{|\psi|} \langle e^{i\phi} f_6 [f_3 - i f_3 (f_4 + f_3 |\psi| \cos \phi) + 3i\beta g_4 |\psi| \cos \phi] \rangle, \quad (41)$$

$$Q'_4 = \frac{1}{|\psi|} \langle e^{i\phi} [i\beta (f_4 + f_3 |\psi| \cos \phi) |\psi| \cos \phi - i\beta f_3 - i\beta |\psi| \sin \phi] \rangle, \quad (42)$$

where Q_4 and Q'_4 defined as $Q_2 + iQ_3$ and $Q'_2 + iQ'_3$.

The dimensionless coefficients Q_n and Q'_n can be evaluated from the solutions of Set A (see OG). We find that these coefficients depend on selecting the model, the amplitude of the warp, the rotation law and the shear viscosity. For a warped β -disc as we consider here, we find three kinds of the internal viscous torques, (see equation (36)):

$$\begin{aligned} G(r, t) = & (Q_1 \mathcal{I} + Q'_1 \Sigma r^2) r^2 \Omega^2 \ell + (Q_2 \mathcal{I} + Q'_2 \Sigma r^2) r^3 \Omega^2 \frac{\partial \ell}{\partial r} \\ & + (Q_3 \mathcal{I} + Q'_3 \Sigma r^2) r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r}. \end{aligned} \quad (43)$$

From a mathematical point of view, the equation (36) constitutes the prototype for a parabolic partial differential equation and can be thought as an advection-diffusion-dispersion equation in the non-linear regime. So for a flat disc ($\frac{\partial \ell}{\partial r} = 0$), the evolution equation reduces to a standard disc diffusion equation.

According to the above equation for $G(r, t)$, the first term on the right hand side gives a contribution to G which is in the local direction ℓ . So, the effective advection coefficients, Q_1 and Q'_1 , represent viscous torques on each ring in the disc due to differential rotation within the disc plane. Thus, the rings tend to rotate in direction ℓ .

Considering neighboring rings in directions ℓ and $\ell + \Delta \ell$, the second term on the right hand side gives a contribution to G which is in the $\frac{\partial \ell}{\partial r}$ direction. Therefore, the effective diffusion coefficients, Q_2 and Q'_2 , represent viscous torques due to the interaction between neighboring rings in the disc, in order to flatten the disc.

The last term on the right-hand side is a dispersion one. So the effective dispersion coefficients Q_3 and Q'_3 demonstrate torques which are perpendicular to ℓ and $\frac{\partial \ell}{\partial r}$, respectively. In this case, each ring in the disc experiences torques tending to make the ring precess if it is not aligned with its neighbors. Then we would expect to generate wave motions in the disc so that the warp propagates as a dispersive wave.

4. Numerical solutions

One can expand the dimensionless functions f_n $\{n = 1, \dots, 6\}$, g_4 and g_5 as power series of the amplitude of the warp to determine the dynamics to any desired order. In Appendix A, truncated Taylor series are presented including the coefficients Q_n and Q'_n . These series enable us to start numerical integration. From the expansions of the dimensionless functions f_n and g_n and their relations to coefficients Q_n and Q'_n , we found truncated Taylor series only for coefficients Q'_1 , Q'_2 and Q_3 . Meanwhile, as we discuss in Appendix A, our solutions are restricted to the nearly Keplerian-discs. So we consider in our calculations for $\tilde{\kappa}^2 = 0.99$ for all values of β .

In order to be able to compare our results with the warped α -disc, we adopt coefficients Q_1 and $Q_4 = Q_2 + iQ_3$ from OG. To distinct between these two coefficients, we label them by superscript α . Now the set of ODEs in the previous section can be solved numerically and we may obtain the coefficients Q'_n and Q_n . In addition, we should note that solutions must satisfy periodic conditions for functions f_n and g_n as well as $\langle f_6 \rangle = 1$ (see, Appendix B). Thus, the derived solutions make an interesting dynamical description of a warped β -disc.

4.1 The warped β -disc

First, we investigate a non-Keplerian disc without viscosity. We consider $\Gamma = 5/3$ and $\beta = \beta_b = 0$. The dimensionless functions f_1, f_2, f_5, f_6 and g_4 are even, but f_3, f_4 and g_5 are odd. So the only non vanishing coefficient is Q_3 . So,

$$\begin{aligned} Q_3 &= \frac{1}{|\psi|} \langle f_6 [f_3 \sin \phi - f_3 \cos \phi (f_4 + f_3 |\psi| \cos \phi) + 3\beta g_4 |\psi| \cos^2 \phi] \rangle |_{\beta=0}, \\ &= \frac{1}{|\psi|} \langle f_6 [f_3 \sin \phi - f_3 \cos \phi (f_4 + f_3 |\psi| \cos \phi)] \rangle. \end{aligned} \quad (44)$$

A contour plot of coefficient Q_3 , is shown in Figure 1.

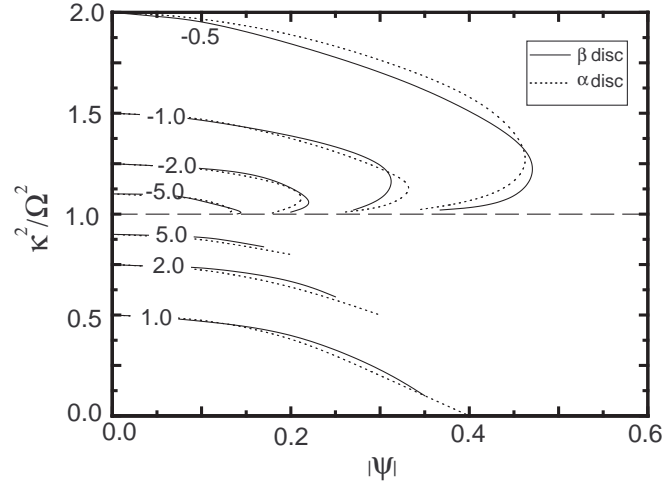


Figure 1. The square of dimensionless epicyclic frequency versus the amplitude of the warp for Contour plots of the coefficients Q_3 (solid line) and Q_3^α (dash line) for an inviscid disc with $\Gamma = 5/3$. The condition $\tilde{\kappa}^2 = 1$ separates solutions into two parts. For $\tilde{\kappa}^2 < 1$, we have $Q_3, Q_3^\alpha > 0$; and for $\tilde{\kappa}^2 > 1$, we see $Q_3, Q_3^\alpha < 0$.

For the case of nearly Keplerian disc with viscosity, we consider $\tilde{\kappa}^2 = 0.99$, $\Gamma = 5/3$ and $\beta_b = 0$. Figure 2, shows β vs. $|\psi|$ and contours of all coefficients. Note that all the plots in Figure 2; parts (a), (c) and (e) show coefficients Q'_n and parts (b), (d) and (f) show coefficients Q_n . The solutions can be calculated for large values of $|\psi|$. For evaluating the coefficients Q_1 and Q'_1 , we used equations (39) and (40). Figures 2a and 2b show contour of plots of the coefficients Q'_1 and Q_1 respectively. For reasonably small values of $|\psi|$ there is a good agreement with the truncated Taylor series for Q'_1 .

Figures 2c and 2d show contour plots of the coefficients Q'_2 and Q_2 respectively. We used equations (41) and (42) for evaluating the coefficients Q_2 and Q'_2 , respectively. For reasonably small values of $|\psi|$ there is a good agreement with the truncated Taylor series for Q'_2 .

Figures 2e and 2f show contours of the coefficients Q'_3 and Q_3 respectively. We used equations (41) and (42) for evaluating the coefficients Q_3 and Q'_3 , respectively. Also Q'_3 is much smaller in magnitude than Q_3 . For reasonably small values of $|\psi|$ there is a good agreement with the truncated Taylor series for Q_3 .

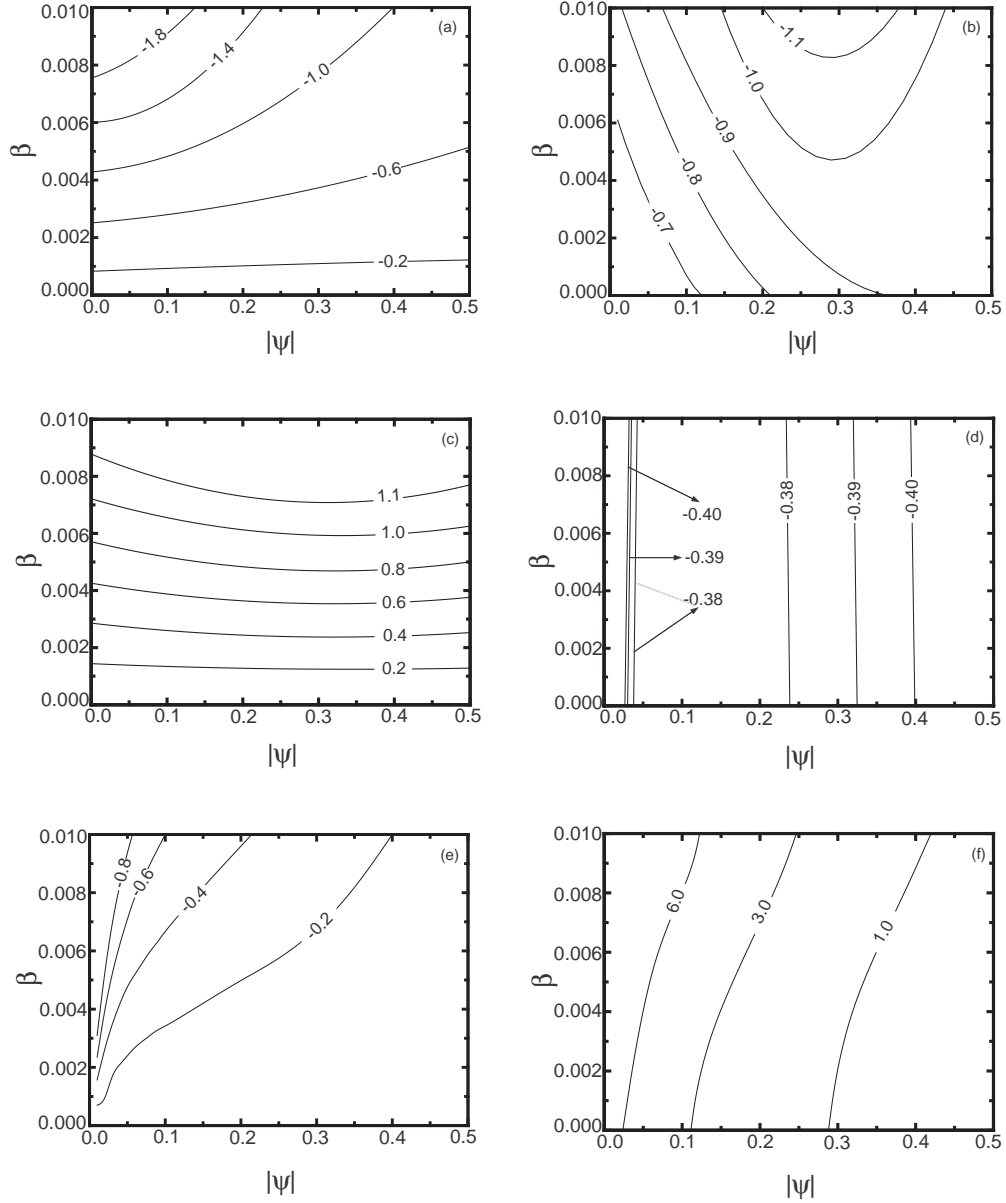


Figure 2. The dimensionless viscosity parameter versus the amplitude of the warp for Contour plots of the coefficients; (a) Q'_1 , (b) Q_1 , (c) Q'_2 , (d) Q_2 , (e) Q'_3 and (f) Q_3 for a viscous, nearly Keplerian disc ($\kappa^2 = 0.99$) with $\Gamma = 5/3$ and $\beta_b = 0$.

4.2 The warped α -disc

First, as an illustrative case, we investigate a non-Keplerian disc without viscosity. We consider $\Gamma = 5/3$ and $\alpha = \alpha_b = 0$. According to obtained functions f_1, f_2, f_5, f_6 by OG, the only non vanishing coefficient is Q_3^α . So,

$$Q_3^\alpha = \frac{1}{|\psi|} \langle f_6[f_3 \sin \phi - f_3 \cos \phi(f_4 + f_3|\psi| \cos \phi)] \rangle. \quad (45)$$

If we plot κ^2/Ω^2 versus $|\psi|$, we get the contours indicated in Figure 1 for Q_3^α . We selected a given interval for $|\psi|$.

For the case of nearly Keplerian disc with viscosity, we consider $\tilde{\kappa}^2 = 0.99$, $\Gamma = 5/3$ and $\alpha_b = 0$. Figure 3 shows α vs. $|\psi|$ and contours of all coefficients. Note that all the plots in Figure 3; parts (a), (b) and (c) show coefficients Q_n^α . We adopt this coefficients from OG,

$$Q_1^\alpha = \langle f_6[\frac{-1}{2}(4 - \tilde{\kappa}^2)\alpha f_2 - f_3 f_5 + \alpha f_2 f_5 |\psi| \cos \phi] \rangle. \quad (46)$$

$$Q_2^\alpha = \frac{1}{|\psi|^2} \langle f_6[(f_4 + f_3|\psi| \cos \phi)(1 + f_3|\psi| \sin \phi) + \alpha f_2 f_3 |\psi| \sin \phi - \alpha f_2(f_4 + f_3|\psi| \cos \phi)|\psi|^2 \cos \phi \sin \phi + \alpha f_2 |\psi|^2 \sin^2 \phi] \rangle. \quad (47)$$

$$Q_3^\alpha = \frac{1}{|\psi|} \langle f_6[f_3 \sin \phi - f_3 \cos \phi(f_4 + f_3|\psi| \cos \phi) + \alpha f_2(f_4 + f_3|\psi| \cos \phi)|\psi| \cos^2 \phi - \alpha f_2 f_3 \cos \phi - \alpha f_2 |\psi| \sin \phi \cos \phi] \rangle. \quad (48)$$

5. Comparing the warped α and β -discs

5.1 Inviscid, non-Keplerian disc

Figure 1 shows a good agreement between two contour plots of Q_3 and Q_3^α . Also they show good agreements with truncated Taylor series for small values of $|\psi|$. As Figure 1 demonstrates the condition $\tilde{\kappa}^2 = 1$ separates solutions into two parts. For $\tilde{\kappa}^2 < 1$, we have $Q_3, Q_3^\alpha > 0$; and for $\tilde{\kappa}^2 > 1$, we see $Q_3, Q_3^\alpha < 0$. In the case $\tilde{\kappa}^2 < 1$ and for small values of $|\psi|$, the physical solutions of β and α models exist only for $|\psi| < 0.35$ and $|\psi| < 0.4$ respectively. The solutions terminates when $g_4 = 0, f_2 \ll 1$ for β model because we couldn't obtain the numerical solution. Whereas the solutions terminate for α model when $f_2 = 0$. Hence, it seems that the stability condition establishes for such discs for all values $\tilde{\kappa}^2 > 0$ in small values of $|\psi|$. In the other word, under the condition $\tilde{\kappa}^2 > 0$, since the epicyclic frequency has the relation with particle orbits, the displacements oscillate about a fixed mean position and the circular orbit is stable to small perturbations.

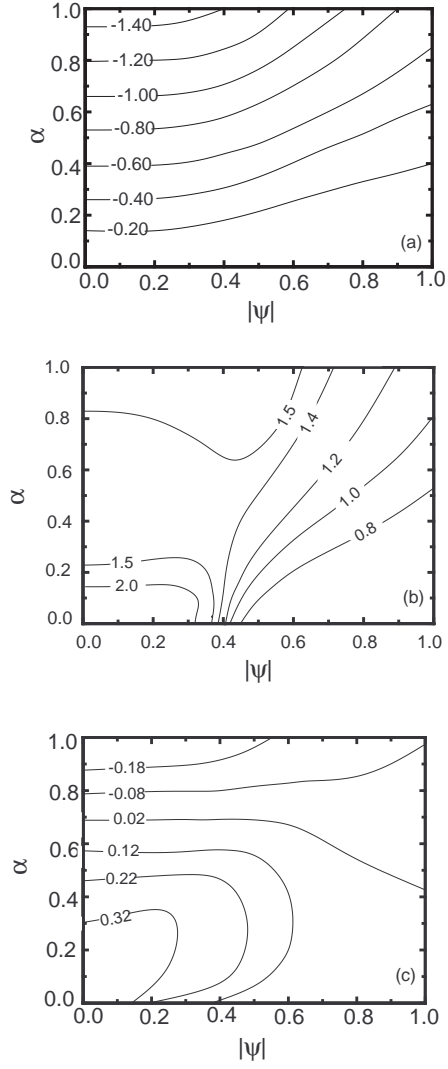


Figure 3. The dimensionless viscosity parameter versus the amplitude of the warp for Contour plots of the coefficients; (a) Q_1^α , (b) Q_2^α and (c) Q_3^α for a viscous, nearly Keplerian disc ($\bar{\kappa}^2 = 0.99$) with $\Gamma = 5/3$ and $\alpha_b = 0$.

5.2 Viscous, nearly Keplerian disc

Now, we compare α and β models for a viscous, nearly Keplerian disc. If we compare the equation (36) with Pringle's equation (130), the coefficients Q'_1 and Q_1 represent viscous torques according to horizontal shear. Figures 2a and 2b show similar behaviors, the magnitudes of Q'_1 and Q_1 increase with increasing $|\psi|$ and this is similar to Q_1^α (Figure 3a). Nevertheless, we see different treatments between two models when torques due to Q'_1 and Q_1 are considered. As mentioned before, they indicate viscous torques parallel to ℓ and Q'_1 , Q_1 identified as the advection coefficients. Then comparing to α model, the advection of the warp occurs more significantly. On the other hand, the viscous stability condition implies that $Q_1, Q'_1 < 0$, so β -discs comparing to α -discs tend to become stable more quickly. This is in agreement to Hure et al. (2001) based on viscous stability for β -discs which is one of the important property of them. Thus, they do not tend to fragment in contrary to α -disc.

Also, the coefficients Q'_2 and Q_2 represent viscous torques due to the vertical shear. Figures 2c and 3b show different behaviors. We see that with increasing $|\psi|$, Q'_2 increases whereas Q_2^α decreases. So, the results due to the torques of Q'_2 and Q_2 don't tend to OG's results. As if, this model affects the vertical structure of α -disc so that viscous torques play more role to flatten β -disc. Also comparing to the OG's analysis, the coefficients of Q'_3 and Q_3 represent torques which lead to the dispersive wave-like propagation of the warp. Figures 2e, 2f and 3c show that Q'_3 , Q_3 and Q_3^α respectively, they treat in a similar manner so that with increasing $|\psi|$, they decrease. Although their typical treatments are similar, Q_3 is typically much larger than Q'_3 and Q_3^α coefficients especially in small $|\psi|$. So we expect to generate wave motion more in small $|\psi|$ for β -disc so that it loses its importance for large $|\psi|$.

6. Summary and Discussion

We have presented an analysis of the non-linear dynamics of a warped accretion disc using β -prescription. Using basic equations of the fluid dynamics in a warped spherical polar coordinates, we obtained the general equations that describe a warped disc. We have also employed the method of matched asymptotic expansions for thin discs to derive a set of coupled PDEs which govern the dynamics of the system. Then we solved the equations by the method of the separation of variables in order to extracting the equations governing the warp in their simplest forms. The non-linear dynamics of a warped β -disc under a differentially rotation field in the presence of a spherically symmetric external potential presents a more complicated behavior comparing to the case in which a warped α -disc is considered (OG). Moreover, it is more complicated than the case of the linear theory of Papaloizou & Pringle (1983) to allow an arbitrary rotation law. In our study, there are six coefficients. They have been determined numerically and analytically, by solving a set of ODEs. Using truncated Taylor series in the amplitude of the warp, the coefficients Q'_1 , Q'_2 and Q_3 are calculated analytically. Figures 1-3, show their contour plots for two

cases inviscid-Keplerian and viscous-nearly Keplerian disc when $\Gamma = 5/3$ and $\beta_b = 0$. We found that these coefficients depend on selecting of the model and the shear viscosity. Considering the equations (39)...(42), one can easily note that the coefficients also depend on Γ , β_b and the value of $|\psi|$. Our results show that the equations governing a warp viscous disc depend on the parameters of the model. The results can be compared with the studies of Pringle (1992) and OG as follows:

1) A comparison with the notation of Pringle (1992), there are four different viscous torques due to the interaction between neighboring rings in the disc (see, Appendix B). Therefore, these torques may be explained by four coefficients. Comparing to OG, we will find two extra coefficients Q'_1 and Q'_2 for the viscous torques. To understand the meaning of these coefficients, we may deduce following equations by comparing equations (130) and (133),

$$Q_1 \mathcal{I} \Omega + Q'_1 \Sigma r^2 \Omega \longleftrightarrow \nu_1 \Sigma \frac{d \ln \Omega}{d \ln r}, \quad (49)$$

$$Q_2 \mathcal{I} \Omega + Q'_2 \Sigma r^2 \Omega \longleftrightarrow \frac{1}{2} \nu_2 \Sigma. \quad (50)$$

We see that the numerical evaluation of the coefficients Q_1, Q'_1 and Q_2, Q'_2 represents the qualitative behavior of ν_1 and ν_2 . We may conclude that ν_1 and ν_2 depend on the value of β .

2) The evolutionary equations for the warped disc show that this scheme is a generalization of the form suggested by OG. A warped accretion disc can be studied in detail by the coefficients Q'_3 and Q_3 in β model and only the coefficient Q_3^α in α model (OG). These coefficients demonstrate torques tending to make the ring precess if it is misaligned with its neighbors. Comparing Figures 2e and 3c show that the direction of the torques is negative for both coefficients Q'_3, Q_3^α . These torques lead to the dispersive wave-like propagation of the warp.

3) Our results show that different viscosity prescriptions and magnitudes (α and β prescriptions) affect dynamics of a warped accretion disc. Therefore, it can be important in determining the viscosity law even for a warped disc.

Some further work still remains to study the dynamics of a viscous warped accretion disc using β -prescription:

a) The dynamics of the warped accretion disc may be studied by considering the magnetic effects. Then we have to modify the equations by including magnetic field of the disc. Thus, one should introduce a suitable model for the geometry of the magnetic field components. Also, one can extend our analysis to the case that self-gravity of the warped discs is important.

b) In this paper, we neglected the thermal and the radiative effects, however, these physical processes are playing important roles in the dynamics of the discs. Therefore, the

behavior of the dynamics of these discs needs to be studied further.

c)As we discussed, the expansions fail only when $\tilde{\kappa}^2 = 1$ for all values of β . This is a resonant case, which can't be described using this method. Therefore, it is interesting to study the non-linear dynamics of the resonant case for a viscous Keplerian (or nearly Keplerian) disc.

References

- Boffin, H. M. J., Stanishev, V., Kraicheva, Z. and Genkov, v. 2003, *ipc, conf*, 297
- Clarkson, W. I., Charles, P. A., Coe, M. J. and Laycock, S. 2003, *Mon. Not. R. Astron. Soc.*, **343**, 1213
- Duschl, W. J., Strittmatter, P. A. and Biermann, P. L. 2000, *Astron. Astrophys.*, **357**, 1123
- Greenhill, L. J., Booth, R. S., Ellingsen, S. P., Herrnstein, J. R., Jauncey, D. L., McCulloch, P. M., Moran, J. M., Norris, R. P., Reynolds, J. E. and Tzioumis, A. K. 2003, *Astrophys. J.*, **590**, 162
- Hure, J. M., Richard, D. and Zahn, J. P. 2001, *Astron. Astrophys.*, **367**, 1087
- Katz, J. I. 1973, *Nature*, **246**, 87
- Lai, D. 1999, *Astrophys. J.*, **524**, 1030
- Lai, D. 2003, *Astrophys. J.*, **591**, L119
- Lubow, S. H. 1992, *Astrophys. J.*, **398**, 525
- Lubow, S.H. and Ogilvie, G.I. 2000, *Astrophys. J.*, **538**, 326
- Lynden-Bell, D. and Pringle, J. 1974, *Mon. Not. R. Astron. Soc.*, **168**, 603
- Mathis, S.,Palacios, A. and Zahn, J. P. 2004, *Astron. Astrophys.*, **425**, 243
- Ogilvie, G. I. 1999, *Mon. Not. R. Astron. Soc.*, **304**, 557 (OG)
- Papaloizou, J. C. B. and Pringle, J. E. 1983, *Mon. Not. R. Astron. Soc.*, **202**, 1181
- Pfeiffer, H. P. and Lai, D. 2004, *Astrophys. J.*, **604**, 766
- Pringle, J. E. 1992, *Mon. Not. R. Astron. Soc.*, **258**, 811
- Pringle, J. E. 1996, *Mon. Not. R. Astron. Soc.*, **281**, 357
- Prinja, R. K., Long, K. S., Froning, C. S., Knigge, C., Witherick, D. K., Clark, J. S. and Ringwald, F. A. 2003, *Mon. Not. R. Astron. Soc.*, **340**, 551
- Quillen, A. C. 2001, *Astrophys. J.*, **563**, 313
- Rau, A., Greiner, J. and McCollough, M. L. 2003, *Astrophys. J.*, **590**, L37
- Richard, D. and Zahn, J. P. 1999, *Astron. Astrophys.*, **347**, 734
- Roberts, W. J. 1974, *Astrophys. J.*, **187**, 575
- Schandl, S. and Meyer, F. 1994, *Astron. Astrophys.*, **289**, 149
- Shakura, N. I. and Sunyaev, R. A. 1973, *Astron. Astrophys.*, **24**, 337
- Still, M. and Boyd, P. 2004, *Astrophys. J.*, **606**, L135
- Tananbaum, H., Gursky, H., Kellogg, E. M., Levinson, R., Schreier, E. and Giacconi, R. 1972, *Astrophys. J.*, **174**, L143

A. Truncated non-linear equations

In this Appendix we find truncated Taylor series of the dimensionless functions f_n $\{n = 1, \dots, 6\}$, g_4 , g_5 and the coefficients Q_1 , Q'_1 , Q_4 and Q'_4 in terms of powers of $|\psi|$. Thus,

$$f_1(\phi) = f_{10} + |\psi|^2 f_{12}(\phi) + O(|\psi|^4), \quad (51)$$

$$f_2(\phi) = f_{20} + |\psi|^2 f_{22}(\phi) + O(|\psi|^4), \quad (52)$$

$$f_3(\phi) = |\psi| f_{31}(\phi) + |\psi|^2 f_{32}(\phi) + |\psi|^3 f_{33}(\phi) + O(|\psi|^4), \quad (53)$$

$$f_4(\phi) = |\psi|^2 f_{42}(\phi) + O(|\psi|^4), \quad (54)$$

$$g_4(\phi) = |\psi| g_{41}(\phi) + |\psi|^3 g_{43}(\phi) + O(|\psi|^5), \quad (55)$$

$$f_5(\phi) = |\psi| f_{51}(\phi) + |\psi|^2 f_{52}(\phi) + |\psi|^3 f_{53}(\phi) + O(|\psi|^4), \quad (56)$$

$$g_5(\phi) = |\psi|^2 g_{52}(\phi) + |\psi|^3 g_{53}(\phi) + O(|\psi|^4), \quad (57)$$

$$f_6(\phi) = f_{60} + |\psi|^2 f_{62}(\phi) + O(|\psi|^4). \quad (58)$$

Note all terms that are scaled with $|\psi|^0$ indicate an unwarped disc.

A.1 Zeroth-order solution

For an unwarped disc, equation (26) at $O(|\psi|^0)$ gives

$$f_{20} = 1. \quad (59)$$

Also equation (33) at $O(|\psi|^0)$ yields

$$f'_{60}(\phi) = 1, \quad (60)$$

and since $\langle f_6 \rangle = 1$ by definition (see, Appendix B), then

$$f_{60} = 1. \quad (61)$$

A.2 First-order solution

The vertical velocity at first order is obtained using equation (27) at $O(|\psi|)$,

$$g'_{41}(\phi) = 0, \quad (62)$$

Hence

$$g_{41} = \tilde{C}_{\theta 1}. \quad (63)$$

Also the horizontal velocities at first order are obtained by equations (25) and (28) at $O(|\psi|)$,

$$f'_{31}(\phi) - 2f_{51}(\phi) = \cos \phi, \quad (64)$$

$$f'_{51}(\phi) + \frac{1}{2}\tilde{\kappa}^2 f_{31}(\phi) = 0. \quad (65)$$

Therefore,

$$f_{31}(\phi) = C_{r1} \cos \phi + S_{r1} \sin \phi, \quad (66)$$

$$f_{51}(\phi) = C_{\phi1} \cos \phi + S_{\phi1} \sin \phi. \quad (67)$$

We can express the solutions using a complex notation as $Z = C + iS$,

$$\begin{bmatrix} -i & -2 \\ \frac{1}{2}\tilde{\kappa}^2 & -i \end{bmatrix} \begin{bmatrix} Z_{r1} \\ Z_{\phi1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (68)$$

The determinant of this matrix is $-(1 - \tilde{\kappa}^2)$, and therefore, there isn't any solution for the Keplerian-disc ($\tilde{\kappa}^2 = 1$). Hence, we have

$$Z_{r1} = \frac{i}{1 - \tilde{\kappa}^2}, \quad (69)$$

$$Z_{\phi1} = \frac{\tilde{\kappa}^2}{2(1 - \tilde{\kappa}^2)}. \quad (70)$$

A.3 Second-order solution

The horizontal velocities at second order are obtained by equations (25), (28) and (29) at $O(|\psi|^2)$

$$f'_{32}(\phi) - 2f_{52}(\phi) = -6(\beta_b + \frac{1}{3}\beta)g_{41} \cos \phi, \quad (71)$$

$$f'_{52}(\phi) + \frac{1}{2}\tilde{\kappa}^2 f_{32}(\phi) = 6\beta g_{52}(\phi), \quad (72)$$

$$g'_{52}(\phi) = g_{41} f_{51}(\phi). \quad (73)$$

Equation (73) yields a solution as

$$g_{52}(\phi) = \tilde{C}_{\theta1} Z_{\phi1} \sin \phi \quad (74)$$

while equations (71) and (72) have the solutions as follows

$$f_{32}(\phi) = C_{r2} \cos \phi + S_{r2} \sin \phi, \quad (75)$$

$$f_{52}(\phi) = C_{\phi2} \cos \phi + S_{\phi2} \sin \phi, \quad (76)$$

with

$$\begin{bmatrix} -i & -2 \\ \frac{1}{2}\tilde{\kappa}^2 & -i \end{bmatrix} \begin{bmatrix} Z_{r1} \\ Z_{\phi1} \end{bmatrix} = \begin{bmatrix} -6(\beta_b + \frac{1}{3}\beta)\tilde{C}_{\theta1} \\ 6i\beta Z_{\phi1}\tilde{C}_{\theta1} \end{bmatrix}. \quad (77)$$

so the solution is

$$Z_{r2} = 6i\tilde{C}_{\theta1} \frac{(\beta_b + \frac{1}{3}\beta) + 2\beta Z_{\phi1}}{\tilde{\kappa}^2 - 1}, \quad (78)$$

$$Z_{\phi2} = 3\tilde{C}_{\theta1} \frac{\tilde{\kappa}^2(\beta_b + \frac{1}{3}\beta) + 2\beta Z_{\phi1}}{\tilde{\kappa}^2 - 1}. \quad (79)$$

Also the enthalpy and vertical velocity at the second order are obtained by equation (23) at $O(|\psi|^2)$,

$$f'_{12}(\phi) = (\Gamma - 1)f_{42}(\phi)f_{10}, \quad (80)$$

equation (24) at $O(|\psi|^2)$,

$$f'_{22}(\phi) = (\Gamma + 1)f_{42}(\phi) - 6(\Gamma - 1)f_{10}g_{41}, \quad (81)$$

and equation (26) at $O(|\psi|^2)$

$$f'_{42}(\phi) = -f'_{31}(\phi) \cos \phi + 2f_{31}(\phi) \sin \phi - f_{22}(\phi). \quad (82)$$

Combining the last two equations, gives

$$f''_{42}(\phi) + (\Gamma + 1)f_{42}(\phi) = 3S_{r1} \sin 2\phi. \quad (83)$$

It has a solution as the following form

$$f_{42}(\phi) = C_{\theta2} \cos 2\phi + S_{\theta2} \sin 2\phi, \quad (84)$$

with

$$\begin{bmatrix} -(3 - \Gamma) & 0 \\ 0 & -(3 - \Gamma) \end{bmatrix} \begin{bmatrix} C_{\theta2} \\ S_{\theta2} \end{bmatrix} = \begin{bmatrix} 0 \\ 3S_{r1} \end{bmatrix}. \quad (85)$$

In a complex notation, the solution is

$$Z_{\theta2} = \frac{3i}{\Gamma - 3} S_{r1}. \quad (86)$$

It then follows that

$$f_{12}(\phi) = -\frac{1}{2}(\Gamma - 1)f_{10}S_{\theta2} \cos 2\phi, \quad (87)$$

$$f_{22}(\phi) = -\frac{1}{2}(\Gamma + 1)S_{\theta2} \cos 2\phi + \frac{\Gamma}{\Gamma - 3} S_{r1}, \quad (88)$$

$$f_{62}(\phi) = S_{\theta2} \cos 2\phi. \quad (89)$$

A.4 Third-order solution

The horizontal velocities at third order are obtained by equations (25), (28) and (29) at $O(|\psi|^3)$,

$$f'_{33}(\phi) - 2f_{53}(\phi) = f_{42}(\phi)f_{31}(\phi) + f_{22}(\phi)\cos\phi, \quad (90)$$

$$f'_{53}(\phi) + \frac{1}{2}\tilde{\kappa}^2 f_{33}(\phi) = f_{42}(\phi)f_{51}(\phi) + 6\beta g_{53}(\phi), \quad (91)$$

$$g'_{53}(\phi) = g_{41}f_{52}(\phi). \quad (92)$$

Equation (92) yields a solution as

$$g_{53}(\phi) = \tilde{C}_{\theta 1} Z_{\phi 2} \sin\phi \quad (93)$$

while equations (90) and (91) have the solutions as follows

$$f_{33}(\phi) = C_{r3} \cos\phi + S_{r3} \sin\phi + \{m = 3 \text{ terms}\}, \quad (94)$$

$$f_{53}(\phi) = C_{\phi 3} \cos\phi + S_{\phi 3} \sin\phi + \{m = 3 \text{ terms}\}. \quad (95)$$

hence, one can obtain

$$Z_{r3} = \frac{i}{\tilde{\kappa}^2 - 1} \left[-\frac{\Gamma}{\Gamma - 3} S_{r1} + \frac{\Gamma + 1}{4} S_{\theta 2} - \frac{1}{2} S_{\theta 2} S_{r1} + S_{\theta 2} Z_{\phi 1} + 12\beta \tilde{C}_{\theta 1} Z_{\phi 2} \right], \quad (96)$$

$$Z_{\phi 3} = \frac{1}{\tilde{\kappa}^2 - 1} \left[\frac{1}{2} S_{\theta 2} C_{\phi 1} + 6\beta \tilde{C}_{\theta 1} Z_{\phi 2} - \frac{\tilde{\kappa}^2}{2} \left(\frac{\Gamma}{\Gamma - 3} S_{r1} - \frac{\Gamma + 1}{4} S_{\theta 2} + \frac{1}{2} S_{\theta 2} S_{r1} \right) \right] \quad (97)$$

The vertical velocity at third order is obtained by equation (27) at $O(|\psi|^3)$

$$g'_{43}(\phi) = 4g_{41}f_{42}(\phi) + g_{41}f_{31}(\phi)g'_{43}(\phi)\cos\phi \quad (98)$$

it yields a solution as

$$g_{43}(\phi) = -2S_{\theta 2}\tilde{C}_{\theta 1} - \frac{1}{4}S_{r1}\tilde{C}_{\theta 1}. \quad (99)$$

A.5 Evaluation of the coefficients

With respect to the functions $f_1 \dots f_6, g_4$ and g_5 , we find truncated Taylor series for the coefficients Q'_1 and Q'_4 as:

$$Q'_1 = Q'_{10} + |\psi|^2 Q'_{12} + O(|\psi|^3), \quad (100)$$

$$Q'_4 = Q'_{40} + |\psi| Q'_{41} + |\psi|^2 Q'_{42} + O(|\psi|^3). \quad (101)$$

At zeroth order, we obtain

$$Q'_{10} = \frac{1}{2}(\tilde{\kappa}^2 - 4)\beta, \quad (102)$$

and

$$\begin{aligned} Q'_{40} &= -i\beta \langle e^{i\phi} (f_{31}(\phi) + \sin \phi) \rangle \\ &= \frac{1}{2}\beta(1 - iZ_{r1}) \\ &= \frac{1}{2}\beta \frac{2 - \tilde{\kappa}^2}{1 - \tilde{\kappa}^2}. \end{aligned} \quad (103)$$

At first order,

$$\begin{aligned} Q'_{41} &= -i\beta \langle e^{i\phi} f_{32}(\phi) \rangle \\ &= \frac{-i\beta}{2} Z_{r2} \\ &= -3\beta \tilde{C}_{\theta 1} \frac{(\beta_b + \frac{1}{3}\beta)(\tilde{\kappa}^2 - 1) + \beta\tilde{\kappa}^2}{(\tilde{\kappa}^2 - 1)^2}. \end{aligned} \quad (104)$$

At second order,

$$\begin{aligned} Q'_{12} &= \frac{1}{2}\beta Z_{\phi 1} \\ &= \frac{\beta}{4} \frac{\tilde{\kappa}^2}{(1 - \tilde{\kappa}^2)}, \end{aligned} \quad (105)$$

and

$$\begin{aligned} Q'_{42} &= i\beta \langle e^{i\phi} (f_{42}(\phi) \cos \phi + f_{31}(\phi) \cos^2 \phi - f_{33}(\phi)) \rangle \\ &= -\frac{i\beta}{2} Z_{r3} \\ &= \frac{\beta}{8(\tilde{\kappa}^2 - 1)^3(\Gamma - 3)} \{4\Gamma(\tilde{\kappa}^2 - 1) - 3(\Gamma + 1)(\tilde{\kappa}^2 - 1) - 6 - 6\tilde{\kappa}^2(\tilde{\kappa}^2 - 1) \\ &\quad + 144(\Gamma - 3)\beta \tilde{C}_{\theta 1}^2 [\tilde{\kappa}^2(\tilde{\kappa}^2 - 1)(\beta_b + \frac{1}{3}\beta) - \beta\tilde{\kappa}^2]\}. \end{aligned} \quad (106)$$

From the coefficients Q_1 and Q_4 , truncated Taylor series were found only for the latter,

$$Q_4 = Q_{40} + |\psi|Q_{41} + |\psi|^2Q_{42} + O(|\psi|^3). \quad (107)$$

At zeroth order,

$$\begin{aligned} Q_{40} &= \langle e^{i\phi} f_{31}(\phi) \rangle \\ &= \frac{1}{2}Z_{r1} \\ &= \frac{i}{2(1 - \tilde{\kappa}^2)}. \end{aligned} \quad (108)$$

At first order,

$$\begin{aligned}
Q_{41} &= \langle e^{i\phi}(f_{32}(\phi) + 3i\beta g_{41} \cos \phi) \rangle \\
&= \frac{i}{2}S_{r2} + \frac{3}{2}i\beta\tilde{C}_{\theta1} \\
&= \frac{3i}{2}\tilde{C}_{\theta1} \frac{3(\beta_b + \frac{1}{3}\beta)(\tilde{\kappa}^2 - 1)^2 + \beta\tilde{\kappa}^2 + \beta(\tilde{\kappa}^2 - 1)^3}{(\tilde{\kappa}^2 - 1)^3}.
\end{aligned} \tag{109}$$

At second order,

$$\begin{aligned}
Q_{42} &= \langle e^{i\phi}(f_{33}(\phi) - if_{31}(\phi)f_{42}(\phi) - if_{31}^2(\phi)\cos \phi) \rangle \\
&= \frac{1}{2}Z_{r3} \\
&= -i\beta Q'_{42}.
\end{aligned} \tag{110}$$

B. Evaluation of the general form of the angular momentum equation

We wish to drive the general form of the angular momentum equation. At first we adopt the equations from OG.²

$$\begin{aligned}
\Sigma \bar{v}_r \frac{\partial}{\partial r}(r^2 \Omega) = & \int \langle \frac{1}{r^2} \partial_r [\mu_0 r^4 \Omega' - \rho_0 r^3 v_{r1}(v_{\phi1} + r v_{r1} \gamma' \cos \beta_E) \\
& + \mu_0 r^3 (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi) \partial_\zeta (v_{\phi1} + r v_{r1} \gamma' \cos \beta_E)] \rangle r d\zeta \\
& - \int \langle \rho_0 [v_{\theta1} + r v_{r1} (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi)] [r \Omega \zeta + r v_{r1} (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi)] \\
& - \mu_0 (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi) \partial_\zeta v_{r1} \\
& + \mu_0 r (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi) (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi) \partial_\zeta [v_{\theta1} \\
& + r v_{r1} (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi)] - \mu_0 r^2 \Omega (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi)^2 \rangle r d\zeta.
\end{aligned} \tag{111}$$

²They can be extracted from the Set B by integration. Note that, the operation $\langle \cdot \rangle$ stands for azimuthally averaged quantities. The range of the integrations with respect to ϕ and ζ are from 0 to 2π and $-\infty$ to ∞ , respectively.

and

$$\begin{aligned}
& \Sigma r^2 \Omega (\dot{\beta}_E + \bar{v}_r \beta'_E) + i \Sigma r^2 \Omega (\dot{\gamma} + \bar{v}_r \gamma') \sin \beta_E = \\
& \int \frac{1}{r^2} (\partial_r + i \gamma' \cos \beta_E) \langle e^{i\phi} \{ \rho_0 r^4 \Omega \zeta v_{r1} - i \rho_0 r^3 v_{r1} \\
& \times [v_{\theta 1} + r v_{r1} (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi)] \\
& + i \mu_0 r^3 (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi) \partial_\zeta [v_{\theta 1} + r v_{r1} (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi)] \\
& - i \mu_0 r^2 \partial_\zeta v_{r1} - i \mu_0 r^4 \Omega (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi) \} \rangle r d\zeta + \int \langle e^{i\phi} \{ -\rho_0 (r^2 \Omega)' \zeta v_{r1} \\
& - \rho_0 (v_{\phi 1} + r v_{r1} \gamma' \cos \beta_E) [v_{\theta 1} + r v_{r1} (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi)] \\
& + i \rho_0 r \Omega \zeta (v_{\phi 1} + r v_{r1} \gamma' \cos \beta_E) \\
& + i \rho_0 r v_{r1} (v_{\phi 1} + r v_{r1} \gamma' \cos \beta_E) (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi) \\
& - \frac{\mu_0}{r} \partial_\zeta (v_{\phi 1} + r v_{r1} \gamma' \cos \beta_E) - i \mu_0 r (\beta'_E \sin \phi - \gamma' \sin \beta_E \cos \phi) \\
& \times [r \Omega' + (\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi) \partial_\zeta (v_{\phi 1} + r v_{r1} \gamma' \cos \beta_E)] \} \rangle r d\zeta.
\end{aligned} \tag{112}$$

where

$$\Sigma \bar{v}_r = \int \langle \rho_0 v_{r2} + \rho_1 v_{r1} \rangle r d\zeta, \quad \Sigma = \int \rho_0 r d\zeta \tag{113}$$

here $\bar{v}_r(r, t)$ and $\Sigma(r, t)$ are the mean radial velocity and the surface density, respectively. To proceed, we define

$$f_6(\phi - \chi) = \tilde{\mathcal{I}}/\mathcal{I} \tag{114}$$

where $\tilde{\mathcal{I}}(r, t)$ is the second vertical moment of the density being defined through

$$\tilde{\mathcal{I}} = \int \rho_0 \zeta^2 r^3 d\zeta \tag{115}$$

so that \mathcal{I} is its azimuthal average, i.e. $\mathcal{I} = \langle \tilde{\mathcal{I}} \rangle$ and also the definition of f_6 requires $\langle f_6 \rangle = 1$. Therefore, by Substituting the relations (18)...(22), (34), (35) and three defined relations (6), (7) and (114) into the equations (111) and (112), the first integral in (111) becomes

$$\begin{aligned}
I_1 = & \frac{1}{r} \partial_r \left[\left\langle \frac{\tilde{\kappa}^2 - 4}{2} r^4 \Omega^2 \beta \right\rangle \Sigma - \langle r^2 \Omega^2 f_3 f_5 f_6 \rangle \mathcal{I} + \langle r^4 \Omega^2 \beta f_5 |\psi| \cos \phi \rangle \Sigma \right. \\
& \left. + 3 \langle r^2 \Omega^2 \beta g_5 f_6 |\psi| \cos \phi \rangle \mathcal{I} \right],
\end{aligned} \tag{116}$$

and the second integral in (111) becomes

$$I_2 = -\langle \Omega^2 f_4 f_6 \rangle \mathcal{I} - \langle \Omega^2 f_3 f_4 f_6 |\psi| \sin \phi \rangle \mathcal{I} - \langle r^2 \Omega^2 \beta f_3 |\psi| \sin \phi \rangle \Sigma$$

$$\begin{aligned}
& + \langle r^2 \Omega^2 \beta f_4 |\psi|^2 \cos \phi \sin \phi \rangle \Sigma + 3 \langle \Omega^2 \beta g_4 f_6 |\psi|^2 \cos \phi \sin \phi \rangle \mathcal{I} \\
& + \langle r^2 \Omega^2 \beta f_3 |\psi|^3 \cos^2 \phi \sin \phi \rangle \Sigma - \langle r^2 \Omega^2 \beta |\psi|^2 \sin^2 \phi \rangle \Sigma \\
& - \langle \Omega^2 f_3 f_6 |\psi| \cos \phi \rangle \mathcal{I} - \langle \Omega^2 f_3^2 f_6 |\psi|^2 \sin \phi \cos \phi \rangle \mathcal{I}.
\end{aligned} \tag{117}$$

similarly, the first integral in (112) [Note that, for any function F, we can write, $\langle e^{i\phi} F(\phi - \chi) \rangle = \langle e^{i(\phi + \chi)} F(\phi) \rangle = \frac{\psi}{|\psi|} \langle e^{i\phi} F(\phi) \rangle$] becomes

$$\begin{aligned}
I'_1 = & \frac{1}{r} (\partial_r + i\gamma' \cos \beta_E) [\langle e^{i\phi} r^2 \Omega^2 (1 - i f_4 - i f_3 |\psi| \cos \phi) f_3 f_6 \rangle \mathcal{I} \frac{\psi}{|\psi|} \\
& + i \langle e^{i\phi} r^4 \Omega^2 \beta f_4 |\psi| \cos \phi \rangle \Sigma \frac{\psi}{|\psi|} + 3i \langle e^{i\phi} r^2 \Omega^2 \beta g_4 f_6 |\psi| \cos \phi \rangle \mathcal{I} \frac{\psi}{|\psi|} \\
& + i \langle e^{i\phi} r^4 \Omega^2 \beta f_3 |\psi|^2 \cos^2 \phi \rangle \Sigma \frac{\psi}{|\psi|} - i \langle e^{i\phi} r^4 \Omega^2 \beta (f_3 + |\psi| \sin \phi) \rangle \Sigma \frac{\psi}{|\psi|}], \tag{118}
\end{aligned}$$

and finally for the second integral in (112), we have

$$\begin{aligned}
I'_2 = & - \langle e^{i\phi} \frac{\tilde{\kappa}^2}{2} \Omega^2 f_3 f_6 \rangle \mathcal{I} \frac{\psi}{|\psi|} - \langle e^{i\phi} \Omega^2 (f_4 f_5 + f_3 f_5 |\psi| \cos \phi) f_6 \rangle \mathcal{I} \frac{\psi}{|\psi|} + i \langle e^{i\phi} \Omega^2 f_5 f_6 \rangle \mathcal{I} \frac{\psi}{|\psi|} \\
& + i \langle e^{i\phi} \Omega^2 f_3 f_5 f_6 |\psi| \sin \phi \rangle \mathcal{I} \frac{\psi}{|\psi|} - \langle e^{i\phi} r^2 \Omega^2 \beta f_5 \rangle \Sigma \frac{\psi}{|\psi|} - 3 \langle e^{i\phi} \Omega^2 \beta f_6 g_5 \rangle \mathcal{I} \frac{\psi}{|\psi|} \\
& - i \langle e^{i\phi} \frac{\tilde{\kappa}^2 - 4}{2} r^2 \Omega^2 \beta |\psi| \sin \phi \rangle \Sigma \frac{\psi}{|\psi|} - i \langle e^{i\phi} r^2 \Omega^2 f_5 \beta |\psi|^2 \sin \phi \cos \phi \rangle \Sigma \frac{\psi}{|\psi|} \\
& - 3i \langle e^{i\phi} \Omega^2 g_5 f_6 \beta |\psi|^2 \sin \phi \cos \phi \rangle \mathcal{I} \frac{\psi}{|\psi|}.
\end{aligned} \tag{119}$$

in which f_n stands for $f_n(\phi)$.

Now, we attempt to arrange them in terms of coefficients Q_1, Q_2, Q'_1, Q'_2 and $Q_4 = Q_2 + iQ_3, Q'_4 = Q'_2 + iQ'_3$. Then, we have

$$I_1 = \frac{1}{r} \partial_r [Q_1 r^2 \Omega^2 \mathcal{I} + Q'_1 r^4 \Omega^2 \Sigma], \tag{120}$$

$$I_2 = -Q_2 \Omega^2 \mathcal{I} |\psi|^2 - Q'_2 r^2 \Omega^2 \Sigma |\psi|^2. \tag{121}$$

and also

$$I'_1 = \frac{1}{r} (\partial_r + i\gamma' \cos \beta_E) (Q_4 r^2 \Omega^2 \mathcal{I} \psi + Q'_4 r^4 \Omega^2 \Sigma \psi), \tag{122}$$

$$I'_2 = Q_1 \mathcal{I} \Omega^2 \psi + Q'_1 \Sigma r^2 \Omega^2 \psi. \tag{123}$$

Therefore, it is possible to get the coefficients Q_n and Q'_n {i.e., $n = 1, \dots, 4$ } in terms of f and g . In the next step, we may write the following combinations,

$$\Sigma \bar{v}_r \frac{\partial}{\partial r} (r^2 \Omega) = I_1 + I_2,$$

$$\begin{aligned}
&= \frac{1}{r} \frac{\partial}{\partial r} (Q_1 \mathcal{I} r^2 \Omega^2) - Q_2 \mathcal{I} r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 + \frac{1}{r} \frac{\partial}{\partial r} (Q'_1 \Sigma r^4 \Omega^2) \\
&\quad - Q'_2 \Sigma r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2,
\end{aligned} \tag{124}$$

$$\begin{aligned}
\Sigma r^2 \Omega \left(\frac{\partial \ell}{\partial t} + \bar{v}_r \frac{\partial \ell}{\partial r} \right) &= I'_1 + I'_2, \\
&= Q_1 \mathcal{I} r \Omega^2 \frac{\partial \ell}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left(Q_2 \mathcal{I} r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right) + Q_2 \mathcal{I} r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 \ell \\
&\quad + \frac{1}{r} \frac{\partial}{\partial r} \left(Q_3 \mathcal{I} r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right) \\
&\quad + Q'_1 \Sigma r^3 \Omega^2 \frac{\partial \ell}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_2 \Sigma r^5 \Omega^2 \frac{\partial \ell}{\partial r} \right) + Q'_2 \Sigma r^4 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 \ell \\
&\quad + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_3 r^5 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right).
\end{aligned} \tag{125}$$

where we used relations (31), (32). Also it has been considered that the disc matter will lie close to $\theta = \pi/2$. So we could write ℓ in terms of spherical polar unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$,

$$\ell = -\mathbf{e}_\theta \Big|_{\theta=\pi/2} \tag{126}$$

hence

$$\frac{\partial \ell}{\partial t} = -\mathbf{e}_r(\dot{\beta}_E \cos \phi + \dot{\gamma} \sin \beta_E \sin \phi) + \mathbf{e}_\phi[\dot{\beta}_E \sin \phi - \dot{\gamma} \sin \beta_E \sin \theta \cos \phi], \tag{127}$$

$$\frac{\partial \ell}{\partial r} = -\mathbf{e}_r(\beta'_E \cos \phi + \gamma' \sin \beta_E \sin \phi) + \mathbf{e}_\phi[\beta'_E \sin \phi - \gamma' \sin \beta_E \sin \theta \cos \phi]. \tag{128}$$

To compare the present work with Pringle's work (1992), we present the following equations (Pringle, 1992),

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma \bar{v}_r) = 0, \tag{129}$$

for the surface density $\Sigma(r, t)$, and

$$\frac{\partial}{\partial t} (\Sigma r^2 \Omega \ell) + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma \bar{v}_r r^3 \Omega \ell) = \frac{1}{r} \frac{\partial}{\partial r} \left(\nu_1 \Sigma r^3 \frac{d\Omega}{dr} \ell \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{2} \nu_2 \Sigma r^3 \Omega \frac{\partial \ell}{\partial r} \right), \tag{130}$$

for the angular momentum $L = \Sigma R^2 \Omega \ell$, in the absence of external torques. Here ν_1 and ν_2 are the viscosity corresponding to the azimuthal and vertical shears, respectively. From these equations, one can derive

$$\Sigma \bar{v}_r \frac{\partial}{\partial r} (r^2 \Omega) = \frac{1}{r} \frac{\partial}{\partial r} \left(\nu_1 \Sigma r^3 \frac{d\Omega}{dr} \right) - \frac{1}{2} \nu_2 \Sigma r^2 \Omega \left| \frac{\partial \ell}{\partial r} \right|^2, \quad (131)$$

for the component of angular momentum parallel to ℓ , and

$$\Sigma r^2 \Omega \left[\frac{\partial \ell}{\partial t} + \left(\bar{v}_r - \nu_1 \frac{d \ln \Omega}{dr} \right) \frac{\partial \ell}{\partial r} \right] = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{2} \nu_2 \Sigma r^3 \Omega \frac{\partial \ell}{\partial r} \right) + \frac{1}{2} \nu_2 \Sigma r^2 \Omega \left| \frac{\partial \ell}{\partial r} \right|^2 \ell, \quad (132)$$

for the tilt vector.

Comparing equations (124), (125) (present work) and (131), (132) (Pringle, 1992), equations (124) and (125) are defined for the component of angular momentum parallel to ℓ and the tilt vector, respectively.

We may simplify the equations (124), (125) and (129) to find the general form of the angular momentum equation as follows

$$\begin{aligned} \frac{\partial}{\partial t} (\Sigma r^2 \Omega \ell) + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma \bar{v}_r r^3 \Omega \ell) = \\ \frac{1}{r} \frac{\partial}{\partial r} (Q_1 \mathcal{I} r^2 \Omega^2 \ell) + \frac{1}{r} \frac{\partial}{\partial r} \left(Q_2 \mathcal{I} r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} (Q_3 \mathcal{I} r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r}) \\ + \frac{1}{r} \frac{\partial}{\partial r} (Q'_1 \Sigma r^4 \Omega^2 \ell) + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_2 \Sigma r^5 \Omega^2 \frac{\partial \ell}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(Q'_3 \Sigma r^5 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right). \end{aligned} \quad (133)$$